TORSION IN PROFINITE COMPLETIONS OF TORSION-FREE GROUPS

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[Received 22 January 1992]

LET G be a residually-finite torsion-free group. Is $\hat{G}$- the profinite completion of G-torsion free? This question was asked in [**CKL**] where it was shown that if G is a finitely generated metabelian-by-finite group then indeed $\hat{G}$ is torsion free. On the other hand Evans [**E**] showed that if G is not finitely generated then it is possible that $\hat{G}$ has torsion. His example is also metabelian. In this note we observe that $\hat{G}$ might have torsion even if G is finitely generated (but of course not metabelian in light of [**CKL**]). In fact we present an example of a finitely generated torsion free group whose pro-finite completion contains as much torsion as one can wish.

THEOREM 1. *There exists a finitely generated, residually finite, torsion-free group G whose pro-finite completion $\hat{G}$ contains an isomorphic copy of every separable profinite group. In particular, $\hat{G}$ contains the cartesian product $\prod_i F_i$ where F_i runs over all (isomorphism classes of) finite groups.*

The proof of Theorem 1 is based on the congruence subgroup problem and some of its related properties (e.g. super-rigidity), but the group G is not a linear group. Still, for linear groups the situation is not much better.

PROPOSITION 2. (a) *There exists a finitely generated torsion-free linear group G, whose profinite completion $\hat{G}$ contains, for every integer $r \geq 2$, uncountably many conjugacy classes of elements of order r.*

(b) *For every finite group F, there exists a finitely generated, torsion free, linear group G whose pro-finite completion contains F^w-the cartesian power of F countably many times.*

Proofs. For $n \geq 3$, the group $\Gamma_n = SL_n(\mathbb{Z})$ has the congruence subgroup property (CSP for short). This means that every finite index subgroup H of Γ_n contains $\Gamma_n(m) = \operatorname{Ker}(SL_n(\mathbb{Z}) \xrightarrow{\varphi_m} SL_n(\mathbb{Z}/m\mathbb{Z}))$ for some non-zero integer m. This is due to Mennicke [**M**] (see [**S**] for an easier proof). In the language of [**BMS**], this means that the pro-finite topology of Γ_n is equal to the congruence topology. As the map φ_m is surjective [**N**, p. 109], this implies that $\widehat{SL_n(\mathbb{Z})} \simeq \varprojlim SL_n(\mathbb{Z}/m\mathbb{Z}) =$

$SL_n(\hat{\mathbb{Z}})$ where $\hat{\mathbb{Z}}$ is the pro-finite completion of the ring of integers. Now, $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ where $\mathbb{Z}_p$ is the ring of p-adic integers and p runs over all primes and $SL_n(\hat{\mathbb{Z}}) = \prod_p SL_n(\mathbb{Z}_p)$. The group Γ_n is finitely generated ([**N**, p. 107]) and clearly residually finite. Hence so are all the finite index subgroups of Γ_n. Γ_n has torsion, but for $m \geq 3$, $\Gamma_n(m)$ is torsion free. Now let G be any finite-index, torsion-free subgroup of Γ_n ($n \geq 3$). The pro-finite completion $\hat{G}$ of G is an open subgroup of $\hat{\Gamma}_n = SL_n(\hat{\mathbb{Z}})$. It therefore contains $\prod_{p \notin S} SL_n(\mathbb{Z}_p)$ as a direct factor where S is a finite set of primes. Now, let r be an integer ≥ 2. By Dirichlet's theorem, there is an infinite set of primes $\mathcal{P}$ with $r \mid p-1$ and $\mathcal{P} \cap S = \emptyset$. For every $p \in \mathcal{P}$, $\mathbb{Z}_p$ has a unit of order r and hence $SL_n(\mathbb{Z}_p)$ has an element ρ_p of order r. For every subset $A \subseteq \mathcal{P}$, we define an element $\alpha_A = (a_p)$ of $SL_n(\hat{\mathbb{Z}})$ by: $a_p = \rho_p$ if $p \in A$ and $a_p = 1$ if $p \notin A$. One can easily check that this gives an element in $\hat{G}$ of order r. If A and A' are different subsets than α_A and α'_A are not conjugate. This proves Proposition 2(a).

For Proposition 2(b): given F choose $n \geq 3$ so that F is a subgroup of $SL_n(\mathbb{Z})$. Then take a torsion-free finite index subgroup G of $SL_n(\mathbb{Z})$ (which in particular avoids F). As before $\hat{G}$ contains $\prod_{p \notin S} SL_n(\mathbb{Z}_p)$ where S is a finite set. Since each $SL_n(\mathbb{Z}_p)$ contains F, $\hat{G}$ contains F^{ω}.

The proof of Theorem 1 is also based on the CSP of $SL_n(\mathbb{Z})$—but this time we need some more properties of these groups. These properties are in fact proved with the help of CSP or by other methods (e.g. Margulis super rigidity cf. [**Ma**] or [**Zi**]). We list them here before starting the proof:

Let $\Gamma_n = SL_n(\mathbb{Z})$ and $n \geq 3$, and let Δ_n be a finite index subgroup of Γ_n. Then

(i) Every normal subgroup of Δ_n is either finite or of finite index [**Ma**, p. 167]), [**S**] or [**Zi**, § 8]).

(ii) let $\varphi_1, \varphi_2\colon \Gamma_n \to SL_{n+1}(\mathbb{Z})$ be the two embeddings: φ_1 embeds Γ_n in the upper left $n \times n$ corner and φ_2 embeds Γ_n in the lower right $n \times n$ corner. Then the group generated by $\varphi_1(\Delta_n)$ and $\varphi_2(\Delta_n)$ is of finite index in Γ_{n+1}. This follows from [V, Cor. 8.3].

(iii) If $\Delta = \prod_{k=3}^{n} \Delta_k$, then there is no epimorphism from Δ onto Δ_{n+1}. This is a consequence of the "super-rigidity" of Δ (cf. [**BMS**, Thm. 16.2], [**Ma**, p. 228], [**S**] or [**Zi**, § 5]) and the fact that there is not epimorphism from $\prod_{k=3}^{n} SL_k(\mathbb{C})$ onto $SL_{n+1}(\mathbb{C})$.

(iv) Let Δ be a subgroup of $H = \prod_{k=3}^{n} SL_k(\mathbb{Z})$. Assume that for every

$k=3,\ldots,n$ the projection of Δ into $SL_k(\mathbb{Z})$ is of finite index in $SL_k(\mathbb{Z})$. Then Δ is of finite index in H. This we prove by induction on n. For $n=3$, there is nothing to prove. For $n \geq 4$, we know by the induction hypothesis that Δ_1, the projection of Δ into $\prod_{k=3}^{n-1} SL_k(\mathbb{Z})$, is of finite index there. Let N be the kernel in Δ of this projection. Then N is a normal subgroup of Δ which is contained in $SL_n(\mathbb{Z})$. It is easy to see that N is also normal in K—the projection of Δ to $SL_n(\mathbb{Z})$. By assumption K is of finite index in $SL_n(\mathbb{Z})$ and so by property (i), N is either finite or of finite index in K. In the second case we are done. In the first case: Let Δ' be a finite index subgroup of Δ such that $\Delta' \cap \mathrm{N} = \{1\}$. Then Δ' is isomorphic on one hand to a finite index subgroup of $\prod_{k=3}^{n-1} SL_k(\mathbb{Z})$ and on the other hand it is mapped onto a finite index subgroup of $SL_n(\mathbb{Z})$. This contradicts (iii).

We are ready now for the proof of Theorem: Denote $K_n = SL_n(\hat{\mathbb{Z}})$ and $K_n(m) = \mathrm{Ker}(SL_n(\hat{\mathbb{Z}}) \to SL_n(\hat{\mathbb{Z}}/m\hat{\mathbb{Z}}))$. Let $K = \prod_{n=3}^{\infty} K_n$ and $\tilde{\Delta}$ a finite index subgroup of $\Gamma_3 = SL_3(\mathbb{Z})$ which is contained in the congruence subgroup $\Gamma_3(m)$ for some $m \geq 3$. Embed i_n: $\Gamma_3 \to SL_n(\hat{\mathbb{Z}})$ in the left upper 3×3 corner and let Δ be the diagonal group $(i_n(\tilde{\Delta}))_{n=3}^{\infty}$. Define an element $\sigma = (\sigma_n)_{n=3}^{\infty}$ of K, where σ_n acts on the standard basis $e_1,\ldots,e_n$ by: $\sigma_n(e_i) = e_{i+1}$ for $i=1,\ldots,n-1$ and $\sigma_n(e_n) = (-1)^n e_1$. Now, let G be the subgroup of K generated Δ and σ. We will prove that G satisfies the properties promised in the theorem.

As Δ is finitely generated G is clearly finitely generated. By induction on n one proves: The group H_n generated by $\langle \Delta, \sigma\Delta\sigma^{-1}, \sigma^2\Delta\sigma^{-2}, \ldots, \sigma^{n-3}\Delta\sigma^{-(n-3)}\rangle$ has the following property: It is mapped injectively onto a finite index subgroup of $\prod_{k=3}^{n} \Gamma_k(m)$.

Indeed, for $n=3$ this is by the definition of Δ. Assume for $n-1$. Then one sees that the projection of H_{n-1} in K_n is inside $\Gamma_{n-1}(m)$ and of finite index there. Thus $\langle H_{n-1} \cup \sigma H_{n-1}\sigma^{-1}\rangle = H_n$ is projected into $\Gamma_n(m)$ and by property (ii), it is of finite index there. Property (iv) now implies that H_n is projected onto a finite index subgroup of $\prod_{k=3}^{n} \Gamma_k(m)$. Moreover, there is no kernel to it as the projections of H_n to Γ_n for $r \geq n$ are coming from the "diagonal" embedding of the projection of H_n to Γ_n.

As $\Gamma_k(m)$ is torsion free for every k, so is H_n, and hence $H = \bigcup_{n=3}^{\infty} H_n$ is torsion free. Let $L_1 = H$ and $L_{n+1} = G^{-1}L_nG$. Then $L_n \subseteq L_{n+1}$ and $L\bigcup_{n=1}^{\infty} L_n$ is also torsion free. L is the normal closure of Δ in G and G/L is an infinite cyclic group generated by σ. This shows that G is also torsion-free.

G being a subgroup of the pro-finite group K is certainly residually finite.

We now claim that in $\hat{G}$ the pro-finite completion of G, the closure $\bar{H}_n$ of H_n is isomorphic to $\hat{H}_n$. This means that the pro-finite topology of G induces on H_n—its pro-finite topology (and not a weaker one as may happen in general). Indeed, by the CSP, the pro-finite topology of H_n is equal to the congruence topology and so $\hat{H}_n$ is equal to the closure of H_n in $\prod_{k=3}^{n} SL_n(\hat{\mathbb{Z}})$. The embedding of G in K induces a map from $\hat{G}$ to K and with respect to this map we see that $\bar{H}_n$ in $\hat{G}$ is mapped to a group isomorphic to $\hat{H}_n$. On the other hand, it is clear that $\bar{H}_n$ is an epimorphic image of $\hat{H}_n$. This implies that $\bar{H}_n \simeq \hat{H}_n$ as both are finitely generated profinite groups. Now, H_n is a finite index subgroup of $\prod_{k=3}^{n} SL_k(\mathbb{Z})$. As in the proof of Proposition 2, we can deduce that $\bar{H}_n$ contains $\prod_{p\in S(n)} SL_n(\hat{\mathbb{Z}}_p)$ where $S(n)$ is a finite set of primes. For every $n \geqslant 3$, choose by induction a prime p_n such that H_n contains $SL_n(\mathbb{Z}_{p_n})$ and $SL_n(\mathbb{Z}_{p_n})$ commute with $SL_k(\mathbb{Z}_{p_k})$ for $k < n$. This way we ensure that $\prod_{n=3}^{\infty} SL_n(\mathbb{Z}_{p_n})$ is a subgroup of $\hat{G}$.

To finish the proof of Theorem 1: Let $A = \prod_{n=3}^{\infty} A_n$ where A_n is the alternating group. $A_n \subseteq SL_n(\mathbb{Z}_{p_n})$ and hence $A \subseteq \hat{G}$. Now (as in [**LW**]), every separable pro-finite group Q is an inverse limit of countably many finite groups and therefore can be embedded in a cartesian product of countably many finite groups. Such a product can be embedded in A and so Q is isomorphic to a subgroup of $\hat{G}$.

Remarks

(1) We do not know if Theorem 1 can be proved with G linear. In fact one may suggest to the contrary, e.g. for every n, there is a number $f(n)$ such that if F is a finite simple group which can be embedded in the pro-finite completion of a linear group of degree n, then F is linear of degree $\leqslant f(n)$ over some field.

(2) We do not know if examples of the kind produced here can be found when the pro-p completion functor is applied on residually-p groups. Here is a weak version of Proposition 2(b) in this context:

PROPOSITION 3. *Given a finite p-group F, there exists a finitely generated, residually-p, torsion free, linear group G whose pro-p completion contains a copy of F.*

Proof. Choose $n \geq 4$ for which $F \subseteq SL_n(\mathbb{Z})$. Choose a prime $q \neq p$, $\mathrm{Ker}(SL_n(\mathbb{Z}_q) \to SL_n(q))$ is a pro-q group and hence F is a subgroup of $SL_n(q)$. Let Q be the pre-image of F in $SL_n(\mathbb{Z}_q)$ and $P = \mathrm{Ker}(SL_n(\mathbb{Z}_p) \to SL_n(p))$. (If $p = 2$ take $P = \mathrm{Ker}(SL_n(\mathbb{Z}_2) \to SL_n(\mathbb{Z}_2/4\mathbb{Z}))$. Then $\prod_{l \neq p,q} SL_n(\mathbb{Z}_l) \times P \times Q$ is isomorphic to an open subgroup of $SL_n(\hat{\mathbb{Z}})$. Hence it is the pro-finite completion of some finite index subgroup Γ of $SL_n(\mathbb{Z})$. Γ is torsion-free and residually-p since $\Gamma \subseteq P$. The pro-p completion of Γ is the maximal pro-p quotient of its pro-finite completion. It is therefore isomorphic to $F \times P$, since $SL_n(\mathbb{Z}_l)$ does not have a non-trivial p-quotient.

(3) A final remark: a dual problem to the one discussed in this paper is: Let G be a finitely generated, residually-finite, infinite torsion group. Is $\hat{G}$ a torsion group? To this question the answer is *always*—NO. This follows from the recent work of Zelmanov [**Z**]. For some cases this was proved before by McMullen [**Mm**].

Acknowledgement

Most of this work was done when the author visited the Universities of Geneve and Neuchatel. The warm hospitality and the support of 3eme Cycle romand de Mathematiques are gratefully acknowledged. We are also grateful to Benjy Weiss: An example of a finitely generated amenable dense subgroup of $\prod A_n$, which we learned from him, was the inspiration for the construction of G of Theorem 1.

REFERENCES

BMS. H. Bass, J. Milnor and J. P. Serre, 'Solution to the Congruence subgroup problems for $SL(n)$ $(n \geq 3)$ and $SP(2n)$ $(n \geq 2)$, *Publ. Math. I.H.E.S.* 33 (1967), 59–137.

CKL. W. W. Crawley-Boevey, P. H. Kropholler and P. A. Linnel, 'Torsion-free soluble groups, completions and the zero divisor conjecture', *J. Pure App. Alg.* 54 (1988), 181–196.

E. M. J. Evans, 'Torsion in pro-finite completions of torsion free groups', *J. Pure App. Alg.* 65 (1990), 101–104.

LW. A. Lubotzky and J. S. Wilson, 'An embedding theorem for profinite groups', *Arch. Math.* 42 (1984), 397–399.

Ma. G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Springer-Verlag 1991.

Me. J. L. Mennicke, 'Finite factor groups of the unimodular group', *Annals of Math.* 81 (1965), 31–37.

Mm. J. R. McMullen, 'The profinite completion of certain residually finite p-groups', *Monatshefte. Math.* 99 (1985), 311–314.

N. M. Newman, *Integral Matrices*, Academic Press, New York, 1972.

S. R. Steinberg, 'Some consequences of the elementary relations in SL_n', *Contemporary Math.* 45 (1985), 335–350 (in: Finite Groups—Coming of Age, ed. J. McKay) A.M.S.-Providence R.I.

V. T. N. Venkataramana, 'Zariski dense subgroups of arithmetic groups', *J. of Alg.* 108 (1987) 325–339.

Zi. R. J. Zimmer, *Ergodic Theory and Semi-simple Groups*, Birkhäuser, Boston 1984.

Z. E. Zelmanov, 'On periodic compact groups', *Israel J. of Math.*, 77 (1992) 83–95.

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